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# A Lyapunov Function for the Robust Second Order Differentiator

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***Abstract*— In this paper we obtain a homogeneous, continuous, quadratic and strict Lyapunov function for Levant's Second Order Differentiator. Since its derivative is a non quadratic, discontinuous, homogeneous form its negative definiteness is determined using some new inequalities, providing coarser bounds than Young's inequalities.**

## I. INTRODUCTION

Differentiators are important in control, as for example in the construction of PID controllers, Unknown Input Observers and in the Fault Detection Problem, to name some applications. Two main problems need to be considered in their construction [7], [8]: the presence of measurement noise, that is inevitable in any application, and the uncertainty in the input signal to be differentiated. The uncertainty of the input signal comes from the fact that the same differentiator is expected to work for a whole class of signals. A differentiator is called *exact* [7] for some input $\sigma(t)$ if it provides the true derivative $\dot{\sigma}(t)$ in the absence of noise. It is called *robust* for $\sigma(t)$ if, in the absence of noise, the output tends uniformly to the true derivative $\dot{\sigma}(t)$ when the input tends uniformly to $\sigma(t)$.

Traditionally, differentiators are constructed by using a linear filter that approximates the transfer function of an ideal differentiator. In particular, linear High-Gain (HG) observers can be used as differentiators [8]. Since no linear (and in fact, nonlinear but continuous) algorithm can be exact on the class of signals with non-vanishing second derivative, i.e. with $L > 0$, HG observers can only approximate an ideal differentiator as its gain tends to infinity.

In [7] A. Levant presented the so-called "Super-Twisting"' (ST) differentiator. This is a discontinuous algorithm that is *exact* and *robust* in the class of signals with bounded second derivative, i.e. $|\ddot{\sigma}(t)| \leq L$, i.e. in absence of noise, it computes exactly the first order derivative of a signal in the class. This algorithm is later generalized to any differentiation order in [9]. Convergence proofs for these differentiators have been obtained mainly by geometric methods [7], [9] and/or using homogeneity properties [10]. For the ST-differentiator a (quadratic) Lyapunov function has been obtained recently [4] (see also [5]), that provide necessary and sufficient conditions for its convergence, and allow to use linear-like analysis and design methods [6], [11]. However, for higher order differentiators such Lyapunov functions have not been obtained.

The objective of this paper is to go one step further in the construction of Lyapunov functions for arbitrary order differentiators and we propose a Lyapunov function for the second order differentiator. This Lyapunov function is a quadratic form but not its derivative, which is also discontinous. This is in contrato to the case of the ST Differentiator [6], [11], for which both the Lyapunov function and its derivative are quadratic forms, so that the determination of their positive and negative definiteness, respectively, can be made with the standard tools of algebra. The main difficulty is determining the negative definiteness of a non quadratic and discontinuous form. Since the use of standard inequalities such as Young's inequality for this purpose seems to be very restrictive we have developed some special inqualities in this paper to solve this problem, which can be seen as generalizations of Young's inequality. The rest of the paper is organized in the following form. In the next section the second order differentiator is recalled, the proposed Lyapunov function and how to determine their positive and negative definiteness, respectively, which consists in a set of nonlinear inequalities in the parameters of the problem. In Section III we provide some details on how to solve this set of inequalities and give some solutions to them. This shows in particular that the inequalities are feasible. In Section IV the main inequalities used in the paper are presented (without proofs, due to lack of space) and, based in them, we provide the proofs of the main results.

**Notation 1.** *in this paper the following notation is used: for a real variable $z \in \mathbb{R}$ to a real power $p \in \mathbb{R}$*

$$z^p = |z|^p \operatorname{sign}(z)$$

*so that* $z^2 = |z|^2 \operatorname{sign}(z)$. *In order to write the usual* $z^2$ *one has to write* $|z|^2$. *Therefore*

$$z^0 = \operatorname{sign}(z)\ ,\ z^p z^q = |z|^p \operatorname{sign}(z)\, |z|^q \operatorname{sign}(z) = |z|^{p+q}$$
$$z^0 z^p = |z|^p\ ,\ z^0 |z|^p = z^p\ .$$

## II. PROBLEM STATEMENT AND MAIN RESULTS

In [9] Levant proposes the 3rd order system

$$\begin{aligned} \dot{z}_1 &= -k_1 \left(z_1 - \sigma(t)\right)^{\frac{2}{3}} + z_2 \\ \dot{z}_2 &= -k_2 \left(z_1 - \sigma(t)\right)^{\frac{1}{3}} + z_3 \\ \dot{z}_3 &= -k_3 \left(z_1 - \sigma(t)\right)^{0} \end{aligned} \tag{1}$$

to estimate exactly and in finite time the first and the 2nd order derivatives of the signal $\sigma(t)$, i.e. $z_2(t) \to \dot{\sigma}(t)$ and $z_3(t) \to \ddot{\sigma}(t)$, for some appropiately designed gains $k_i$, $i = 1, 2, 3$. Defining the differentiation errors $x_i \triangleq z_i - \sigma^{(i)}(t)$

their dynamics is given by

$$\begin{aligned}\dot{x}_1 &= -k_1 x_1^{\frac{2}{3}} + x_2\\ \dot{x}_2 &= -k_2 x_1^{\frac{1}{3}} + x_3\\ \dot{x}_3 &= -k_3 x_1^{0} + \pi(t)\end{aligned} \tag{2}$$

where $\pi(t) = \sigma^{(3)}(t)$ corresponds to the third order derivative of the signal $\sigma(t)$, that is assumed to be uniformly bounded, i.e. for all $t \geq 0$, $|\pi(t)| \leq \Delta$. Note that (2) is homogeneous [1], [10] of degree $\delta_f = -1$ with weights $\varrho = [3, 2, 1]$, and its solutions are understood in the sense of Filippov [2]. Recall that a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ (or a differential inclusion) is called *homogeneous of degree* $\delta \in \mathbb{R}$ *with the dilatation* $d_\kappa : (x_1, x_2, \ldots, x_n) \mapsto (\kappa^{\rho_1} x_1, \kappa^{\rho_2} x_2, \ldots, \kappa^{\rho_n} x_n)$, where $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ are some positive numbers (called the *weights*), if for any $\kappa > 0$ the identity holds $f(x) = \kappa^{-\delta} d_\kappa^{-1} f(d_\kappa x)$. A scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous of degree* $\delta \in \mathbb{R}$ *with the dilatation* $d_\kappa$ if for any $\kappa > 0$ the identity holds $V(x) = \kappa^{-\delta} V(d_\kappa x)$.

Consider the following continuous Function

$$\begin{aligned}V(x) = \gamma_1 |x_1|^{\frac{4}{3}} - \gamma_{12} x_1^{\frac{2}{3}} x_2 + \gamma_2 |x_2|^2 + \gamma_{13} x_1 x_3\\ - \gamma_{23} x_2 x_3^2 + \gamma_3 |x_3|^4 .\end{aligned}$$

$V(x)$ is homogeneous [1] of degree $\delta_V = 4$, with weights $\varrho = [3, 2, 1]$, it is differentiable almost everywhere, but due to the term $x_1^{\frac{2}{3}}$ it is not locally Lipschitz. We will derive conditions for the coefficients $(\gamma_1, \gamma_{12}, \gamma_2, \gamma_{13}, \gamma_{23}, \gamma_3)$ and for the gains $(k_1, k_2, k_3)$ of the algorithm such that $V(x) > 0$ and $\dot{V} < 0$ for all $x \in \mathbb{R}^3$, $x \neq 0$. For simplicity, we will consider here the case in which $\gamma_{13} = 0$. In that case $V(x)$ is a quadratic form in the vector $\xi^T = \left[\begin{array}{ccc} x_1^{\frac{2}{3}}, & x_2, & x_3^2 \end{array}\right]$, i.e.

$$V(x) = \xi^T \Gamma \xi, \; \Gamma = \begin{bmatrix} \gamma_1 & -\frac{1}{2}\gamma_{12} & 0\\ -\frac{1}{2}\gamma_{12} & \gamma_2 & -\frac{1}{2}\gamma_{23}\\ 0 & -\frac{1}{2}\gamma_{23} & \gamma_3 \end{bmatrix}. \tag{3}$$

This implies that that $V(x)$ is p.d. if and only if $\Gamma > 0$. However, $\dot{V}$ is not a quadratic form.

To state the main results we will introduce two relevant functions.

- Function $b(a;\rho)$ is a real valued function of the real, non negative variable $a \geq 0$, with a (real) parameter $\rho$ given by

$$b(a;\rho) = \frac{4}{6^{\frac{3}{2}} a^{\frac{1}{2}}} \frac{12a - |\rho|\left(|\rho| + \sqrt{|\rho|^2 + 12a}\right)}{\left(|\rho| + \sqrt{|\rho|^2 + 12a}\right)^{\frac{1}{2}}}. \tag{4}$$

  Note that $\frac{\partial b(a;\rho)}{\partial a} > 0$ so that $b(a;\rho)$ is monotonically increasing (in $a$). Moreover, $b\left(\frac{1}{4}|\rho|^2;\rho\right) = 0$ and therefore $b(a;\rho)$ is negative for $a \in \left(0, \frac{1}{4}|\rho|^2\right)$ and it is positive for $a \in \left(\frac{1}{4}|\rho|^2, \infty\right)$. Furthermore,

$$\lim_{a \to 0^+} b(a;\rho) = -\infty .$$

- Function $\beta(\alpha;\lambda,\mu(t))$ is a real valued function of the non negative real variable $\alpha \geq 0$ and depending on a real parameter $\lambda$ and a real, non negative, time-varying function $\mu(t)$, that is uniformly bounded

$$0 \leq \mu_m \leq \mu(t) \leq \mu_M \;\; \forall t \geq 0 . \tag{5}$$

$$\begin{aligned}&\beta(\alpha;\lambda,\mu(t)) =\\ &= \begin{cases} \max\left(\beta_1(\alpha) - \mu_m, \beta_2(\alpha) + \mu_M\right), & \alpha \leq \frac{1}{3}|\lambda|^2\\ \max\left(\beta_1(\alpha) - \mu_m, \mu_M\right), & \alpha > \frac{1}{3}|\lambda|^2 \end{cases},\end{aligned} \tag{6}$$

  where

$$\begin{aligned}\beta_1(\alpha) &= -\alpha r_1^3(\alpha) - \lambda r_1^2(\alpha) + r_1(\alpha)\\ \beta_2(\alpha) &= -\alpha r_2^3(\alpha) + \lambda r_2^2(\alpha) - r_2(\alpha) .\end{aligned} \tag{7}$$

  and

$$r_1(\alpha) = \frac{-\lambda + \sqrt{|\lambda|^2 + 3\alpha}}{3\alpha}, \; r_2(\alpha) = \frac{\lambda + \sqrt{|\lambda|^2 - 3\alpha}}{3\alpha} .$$

The derivative of the Lyapunov function candidate (3) along the trajectories of the perturbed system is (almost everywhere)

$$\begin{aligned}\dot{V}(x) = -q_{11}|x_1| + q_{12} x_1^{\frac{1}{3}} x_2 - \frac{2}{3}\gamma_{12}\frac{|x_2|^2}{|x_1|^{\frac{1}{3}}} - \gamma_{12} x_1^{\frac{2}{3}} x_3\\ + \gamma_{23} k_2 x_1^{\frac{1}{3}} x_3^2 - q_{13} x_1^0 x_3^3 + q_{23} x_2 x_3 - \gamma_{23}|x_3|^3 .\end{aligned} \tag{8}$$

$$\begin{aligned}q_{11} &= \left(\frac{4}{3}\gamma_1 k_1 - \gamma_{12} k_2\right), \; q_{13} = 4\gamma_3\left(k_3 - x_1^0 \pi(t)\right)\\ q_{12} &= 2\left(\frac{2}{3}\gamma_1 - \gamma_2 k_2 + \frac{1}{3}\gamma_{12} k_1\right)\\ q_{23} &= 2\left[\gamma_2 + \gamma_{23}\left(k_3 - x_1^0 \pi(t)\right) x_1^0 x_3^0\right]\end{aligned}$$

Note that $\dot{V}$ is homogeneous of degree $\delta_{\dot{V}} = 3$, with weights $\varrho = [3, 2, 1]$, and it is discontinuous.

The main result of the paper is the following Theorem

**Theorem 1.** *Consider the continuous and homogeneous function* $V(x)$ *given by* (3)*.* $V(x)$ *is positive definite and radially unbounded if and only if*

$$\begin{aligned}\gamma_1 > 0, \; \gamma_1\gamma_2 - \tfrac{1}{4}\gamma_{12}^2 &> 0\\ \left[\gamma_1\gamma_2 - \tfrac{1}{4}\gamma_{12}^2\right]\gamma_3 - \tfrac{1}{4}\gamma_1\gamma_{23}^2 &> 0 .\end{aligned} \tag{9}$$

$\dot{V}$ *given by* (8) *is negative definite for every value of the perturbation satisfying* $|\pi(t)| \leq \Delta$ *if*

$$k_3 > \Delta , \tag{10}$$

$$\gamma_{12} > 0 \tag{11}$$

$$\gamma_{23} > 0 \tag{12}$$

*and there exists an $\alpha > 0$ such that*

$$b\left(z\left(\alpha\right);\rho\right) > \sqrt{\frac{2^4}{3^2}\frac{\left(\left|\gamma_2\right| + \left|\gamma_{23}\left(k_3+\Delta\right)\right|\right)^3}{\gamma_{12}\gamma_{23}\left(1-k_2\beta\left(\alpha;\lambda,\mu\left(t\right)\right)\right)}} \quad (13)$$

$$z\left(\alpha\right) > \frac{1}{4}\left|\rho\right|^2 \quad (14)$$

$$\beta\left(\alpha;\lambda,\mu\left(t\right)\right) < \frac{1}{k_2} \quad (15)$$

*where*

$$z\left(\alpha\right) \triangleq \frac{2}{3}\left(\frac{4}{3}\gamma_1 k_1 - \gamma_{12}k_2 - \gamma_{23}k_2\alpha\right)\gamma_{12}\,,$$

$$\rho = q_{12} = 2\left(\frac{2}{3}\gamma_1 - \gamma_2 k_2 + \frac{1}{3}\gamma_{12}k_1\right)\,,\ \lambda = \frac{\gamma_{12}}{\gamma_{23}k_2}\,, \quad (16)$$

*and*

$$\mu_m \triangleq 4\frac{\gamma_3\left(k_3-\Delta\right)}{\gamma_{23}k_2}\,,\ \mu_M \triangleq 4\frac{\gamma_3\left(k_3+\Delta\right)}{\gamma_{23}k_2} \quad (17)$$

$$\mu\left(t\right) = 4\frac{\gamma_3\left(k_3 - x_1^0\pi\left(t\right)\right)}{\gamma_{23}k_2}\,.$$

*In this case $V\left(x\right)$ satisfies the differential inequality*

$$\dot{V} \leq -\kappa V^{\frac{3}{4}} \quad (18)$$

*for some positive $\kappa$ and it is a Lyapunov function for the system* (2)*, whose trajectories converge in finite time to the origin $x = 0$ for every value of the perturbation. The convergence time of a trajectory starting at the initial condition $x_0$ can be estimated as*

$$T\left(x_0\right) \leq \frac{4}{\kappa}V^{\frac{1}{4}}\left(x_0\right)\,. \quad (19)$$

The unperturbed case is obtained by setting $\pi\left(t\right) = 0$ and $\Delta = 0$. In particular, it is obtained that $\mu_m = \mu_M = \mu = 4\frac{\gamma_3 k_3}{\gamma_{23}k_2}$.

The previous Theorem provides conditions to find a Lyapunov function for the system (2). However, it is not a priori obvious that there exist indeed values of the parameters $(k_1, k_2, k_3, \gamma_1, \gamma_2, \gamma_3, \gamma_{12}, \gamma_{23}, \alpha)$ for which the conditions imposed in the Theorem are satisfied, i.e. if the system of inequalities is feasible. In the next section it will be shown that there are indeed sets of values for the parameters, that fulfill the conditions of the Theorem.

**Corollary 2.** *Suppose that the origin $x = 0$ is finite-time stable for system* (2) *a set of gains $(k_1,\, k_2,\, k_3)$, and that $V\left(x\right)$ in* (3) *is a Lyapunov function for a set of parameters $(\gamma_1,\, \gamma_{12},\, \gamma_2,\, \gamma_{23},\, \gamma_3)$. Then the origin $x = 0$ is also finite-time stable for system* (2) *for the set of gains $\left(Lk_1,\, L^2k_2,\, L^3k_3\right)$, and that $V\left(x\right)$ in* (3) *is a Lyapunov function for the set of parameters $\left(L^{-4}\gamma_1,\, L^{-5}\gamma_{12},\, L^{-6}\gamma_2,\, L^{-9}\gamma_{23},\, L^{-12}\gamma_3\right)$, for every constant real number satisfying $L > 0$ in the unperturbed case ($\Delta = 0$), or $L \geq 1$ in the perturbed case ($\Delta \neq 0$).*

## III. Feasibility of the System of Inequalities

In order to find values of the gains $(k_1, k_2, k_3)$ that render stable the differentiator (1) stable and to find a Lyapunov function for it, i.e. find the parameters $(\gamma_1, \gamma_2, \gamma_3, \gamma_{12}, \gamma_{23})$, the highly nonlinear system of inequalities (9), (10)-(12) and (13)-(15) have to be solved. Moreover, it is not obvious that this system of inequalities is feasible.

Given a value of $\Delta$, a set of values for $(k_1, k_2, k_3, \gamma_1, \gamma_2, \gamma_3, \gamma_{12}, \gamma_{23})$ is *feasible* if it satisfies the set of inequalities. This can be established by checking directly the inequalities (9), (10)-(12). The last three ones (13)-(15) require to find a value of $\alpha > 0$ fulfilling the conditions.

Selecting $\gamma_1 = 1$ (without loss of generality), $(\gamma_{12},\, \gamma_{23},\, k_3)$ satisfying (10)-(12) and

$$\gamma_2 = \frac{1}{4}\eta\gamma_{12}^2\,,\ \eta > 1\,,\ \gamma_3 = \frac{\gamma_{23}^2}{\left(\eta-1\right)\gamma_{12}^2}\delta\,,\ \delta > 1\,,$$

inequalities (9) will be satisfied for every $\eta > 1$ and $\delta > 1$. Replacing these expressions in (13) and taking the square power one obtains

$$\beta\left(\alpha;\lambda,\mu\left(t\right)\right) < \nu\left(\alpha\right) \quad (20)$$

$$\nu\left(\alpha\right) \triangleq \frac{1}{k_2}\left[1 - \frac{2^4}{3^2}\frac{\left(\frac{1}{4}\eta\gamma_{12}^2 + \gamma_{23}\left(k_3+\Delta\right)\right)^3}{\gamma_{12}\gamma_{23}b^2\left(z\left(\alpha\right);\rho\right)}\right]$$

where

$$b^2\left(a;\rho\right) = \frac{2}{3^3 a}\left[\frac{3^2\cdot 2^4 a^2}{\left(\left|\rho\right| + \sqrt{\left|\rho\right|^2 + 12a}\right)} + \right.$$
$$\left. + \left|\rho\right|^2\left(\left|\rho\right| + \sqrt{\left|\rho\right|^2+12a}\right)\right] - \frac{2^4}{3^2}\left|\rho\right|\,,$$
$$\rho = 2\left(\frac{2}{3} - \frac{1}{4}\eta\gamma_{12}^2k_2 + \frac{1}{3}\gamma_{12}k_1\right)\,.$$

Condition (14) corresponds to an upper bound for $\alpha$, i.e.

$$\alpha < \frac{4}{3}\frac{\gamma_1 k_1}{\gamma_{23}k_2} - \frac{\gamma_{12}}{\gamma_{23}} - \frac{3}{2\gamma_{12}\gamma_{23}k_2}\left(\frac{2}{3} - \frac{1}{4}\eta\gamma_{12}^2k_2 + \frac{1}{3}\gamma_{12}k_1\right)^2\,,$$

and condition (15) clearly corresponds to an upper bound for $\beta$. Within these bounds for $\alpha$ and $\beta$ we have to check if inequality (20) is fulfilled. This inequality has a simple graphical interpretation: the graph of the function $\beta\left(\alpha\right)$ has to go below the graph of function $\nu\left(\alpha\right)$ for some value of $\alpha$ in the permitted interval. In the next subsection we will use this procedure to determine some feasible values for $(k_1, k_2, k_3, \gamma_1, \gamma_2, \gamma_3, \gamma_{12}, \gamma_{23})$.

*Remark* 3. In the special case when $\rho = 0$ then

$$b^2\left(a\right) = \frac{2^4}{3^{\frac{3}{2}}}a^{\frac{1}{2}}$$
$$k_2 = \frac{4}{3}\frac{1}{\eta\gamma_{12}}\left(\frac{2}{\gamma_{12}} + k_1\right) \quad (21)$$

| | Fig. 1 | Fig. 2 | Fig. 3 |
|---|---|---|---|
| $k_1$ | 1.2 | 4.44 | 13.2 |
| $k_2$ | 0.42 | 5.75 | 50.82 |
| $k_3$ | 0.01 | 0.5 | 13.31 |
| $\gamma_1$ | 1 | 1 | 1 |
| $\gamma_2$ | 4.17 | 0.3 | 0.034 |
| $\gamma_3$ | 323.22 | $9 \times 10^{-3}$ | $1.5 \times 10^{-6}$ |
| $\gamma_{12}$ | 2.71 | 0.73 | 0.246 |
| $\gamma_{23}$ | 31.7 | 0.046 | $1.97 \times 10^{-4}$ |
| $\Delta$ | 0 | 0.1 | 0.1 |
| $\eta$ | 2.27 | 2.27 | 2.27 |
| $\delta$ | 3 | 3 | 3 |
| $\mu_m$ | 0.97 | 0.057 | $7.9 \times 10^{-3}$ |
| $\mu_M$ | 0.97 | 0.085 | $8.1 \times 10^{-3}$ |

TABLE I
PARAMETER VALUES

and therefore

$$\beta\left(\alpha;\lambda,\mu\left(t\right)\right) < \frac{3\eta\gamma_{12}}{4\left(\frac{2}{\gamma_{12}}+k_1\right)}\left[1- -\frac{3^{\frac{1}{2}}}{2^{\frac{3}{2}}}\frac{\left(\frac{1}{4}\eta\gamma_{12}^2+\gamma_{23}\left(k_3+\Delta\right)\right)^3}{\gamma_{12}^{\frac{3}{2}}\gamma_{23}^{\frac{3}{2}}k_2^{\frac{1}{2}}\sqrt{\frac{1}{\eta\gamma_{23}k_2}\left(\left(\eta-1\right)k_1-\frac{2}{\gamma_{12}}\right)-\alpha}}\right]$$

### A. Some feasible values for $(k_1, k_2, k_3, \gamma_1, \gamma_2, \gamma_3, \gamma_{12}, \gamma_{23})$

We first consider $\Delta = 0$, $\gamma_1 = 1$, and select $k_2$ as in (21), so that $\rho = q_{12} = 0$. Selecting the values for $(k_1, k_3, \gamma_{12}, \gamma_{23}, \nu, \delta)$ given in the second column of Table I, the other values of the parameters are calculated as described previously. The graphs of functions $\nu(\alpha)$ and $\beta(\alpha)$ are shown in Figure 1. Note that inequality (20) is fulfilled for these values of the parameters, since in the interval (approx.) $\alpha \in [0.011\,,\,0.014]$ the graph of $\beta$ is below the graph of $\nu$. In the same manner it is possible to check that selecting the same values of $(k_3, \gamma_{12}, \gamma_{23}, \nu, \delta)$ given in second column of Table I, and for every value of $k_1$ in the interval $k_1 \in [1.1\,,\,2.4]$ the inequalities are still satisfied. The same is true if one selects the same values of $(k_1, \gamma_{12}, \gamma_{23}, \nu, \delta)$ given in the second column of Table I, and $k_3 \in [0.004\,,\,0.016]$. Note that according to Corollary 2 all appropiately scaled values of $(k_1, k_2, k_3, \gamma_1, \gamma_2, \gamma_3, \gamma_{12}, \gamma_{23})$ are also feasible for any $L > 0$.

Now we consider $\Delta = 0.1$, $\gamma_1 = 1$, and select $k_2$ as in (21), so that $\rho = q_{12} = 0$. Selecting the values for $(k_1, k_3, \gamma_{12}, \gamma_{23}, \nu, \delta)$ given in the third column of Table I, the other values of the parameters are calculated as described previously. The graphs of functions $\nu(\alpha)$ and $\beta(\alpha)$ are shown in Figure 2. Inequality (20) is fulfilled for these values of the parameters, since in the interval (approx.) $\alpha \in [2.1\,,\,2.4]$ the graph of $\beta$ is below the graph of $\nu$. Note that according to Corollary 2 all appropiately scaled values of $(k_1, k_2, k_3, \gamma_1, \gamma_2, \gamma_3, \gamma_{12}, \gamma_{23})$ are also feasible for any $L > 1$. In particular, using a value for $L = 2.973$ the values in the fourth column of Table I are obtained. Figure 3 shows the graphs of functions $\nu(\alpha)$ and $\beta(\alpha)$ in this case, confirming

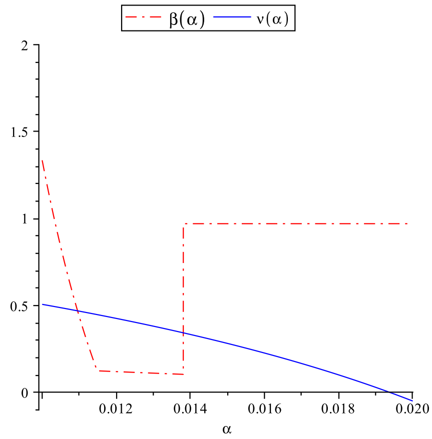


Fig. 1. Graphs of the two functions involved in inequality (20) with values in the second column of Table I

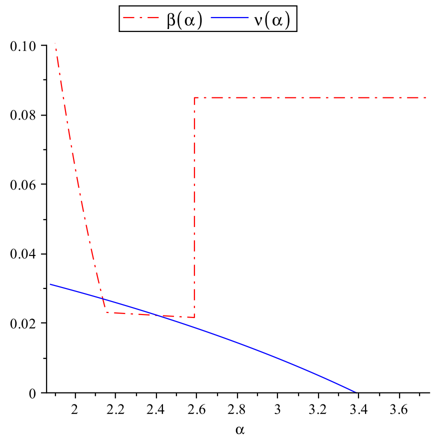


Fig. 2. Graphs of the two functions involved in inequality (20) with values in the third column of Table I

that the inequality (20) is fulfilled for these values of the parameters, since in the interval (approx.) $\alpha \in [160\,,\,205]$ the graph of $\beta$ is below the graph of $\nu$.

## IV. PROOF OF THE RESULTS

In this section we will give a sketch of the proof of the main results.

Note that the positive definiteness of $V(x)$ in (3) is easy to check: $V(x)$ is p.d. if and only if $\Gamma > 0$, i.e. if and only if (9) is fulfilled.

Now we will consider the derivative of $\dot{V}(x)$ along the trajectories of the perturbed system (8), since the unperturbed case will be a particular case of it.

### A. Some inequalities

For the proof of the negative definiteness of $\dot{V}$ we will require some inequalities. We will peresent them in this subsection without a proof, due to lack of space.

From the classical arithmetic and geometric means inequality [3, Thm. 9, Section 2.5] it follows the classical Young's inequality:

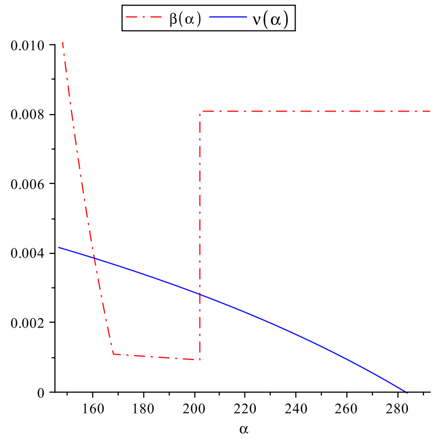


Fig. 3. Graphs of the two functions involved in inequality (20) with values in the fourth column of Table I

**Lemma 4.** *For every real numbers $a > 0$, $b > 0$, $c > 0$, $p > 1$, $q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$ the following inequality is satisfied*

$$ab \leq c^p \frac{a^p}{p} + c^{-q} \frac{b^q}{q} \ . \tag{22}$$

It follows from here that

$$|x_1|^{\frac{2}{3}} |x_3| \leq \frac{2\beta_{13}^{\frac{3}{2}}}{3} |x_1| + \frac{\beta_{13}^{-3}}{3} |x_3|^3 \tag{23}$$

$$|x_1|^{\frac{1}{3}} |x_3|^2 \leq \frac{\eta_{13}^3}{3} |x_1| + \frac{2\eta_{13}^{-\frac{3}{2}}}{3} |x_3|^3 \tag{24}$$

$$|x_2| |x_3| \leq \frac{2\beta_{23}^{\frac{3}{2}}}{3} |x_2|^{\frac{3}{2}} + \frac{\beta_{23}^{-3}}{3} |x_3|^3 \tag{25}$$

The following inequalities are derived for our particular problem.

**Lemma 5.** *Consider a non negative, time-varying $\mu(t) \geq 0$, satisfying*

$$0 \leq \mu_m \leq \mu(t) \leq \mu_M \ . \tag{26}$$

*Then, for every real value of $x_1$ and $x_3$, any real value of $\lambda$, and any positive value of $\delta > 0$ the inequality*

$$-\lambda x_1^{\frac{2}{3}} x_3 + \delta x_1^{\frac{1}{3}} x_3^2 - \mu(t)\, \delta^2 x_1^0 x_3^3 \leq \frac{\alpha}{\delta} |x_1| + \vartheta \delta^2 |x_3|^3 \tag{27}$$

*is satisfied for all t, and every possible signal $\mu(t)$ satisfying 26, if and only if*

$$\vartheta \geq \beta(\alpha; \lambda, \mu(t))$$

*where the function $\beta(\alpha; \lambda, \mu(t))$ is defined in* (6).

The previous Lemma generalizes Young's inequality, when several monomials are considered. Applying Young's lemma to each of the monomials in (27) one can obtain a similar inequality, i.e. using (23) and (24) one obtains

$$-\lambda x_1^{\frac{2}{3}} x_3 + \delta x_1^{\frac{1}{3}} x_3^2 - \mu(t)\, \delta^2 x_1^0 x_3^3 \leq \tilde{\alpha} |x_1| + \tilde{\beta} |x_3|^3 \ ,$$

where

$$\tilde{\alpha} = \left( \frac{2\beta_{13}^{\frac{3}{2}}}{3} |\lambda| + \frac{\eta_{13}^3}{3} \delta \right) ,\ \tilde{\beta} = \left( \frac{\beta_{13}^{-3}}{3} |\lambda| + \frac{2\eta_{13}^{-\frac{3}{2}}}{3} \delta + \mu_M \delta^2 \right) .$$

However, this last inequality is in general much more conservative than (27), that is tight, since its conditions are necessary and sufficient. Moreover, when (27) has only one monomial on the left hand side, then it reduces to Young's inequality. To see this, consider the case when $\lambda = 0$, $\delta = 1$ and $\mu(t) = 0$ in (27). Then we obtain that

$$\beta(\alpha) \geq \beta_1(\alpha) = -\alpha \left( \frac{1}{\sqrt{3\alpha}} \right)^3 + \left( \frac{1}{\sqrt{3\alpha}} \right) = \frac{2}{3\sqrt{3\alpha}}$$

i.e.

$$x_1^{\frac{1}{3}} x_3^2 \leq \alpha |x_1| + \beta(\alpha) |x_3|^3 \ .$$

By Young's inequality one obtains in this case

$$x_1^{\frac{1}{3}} x_3^2 \leq \frac{\gamma^3}{3} |x_1| + \frac{2\gamma^{-\frac{3}{2}}}{3} |x_3|^3$$

and so

$$\beta(\alpha) = \frac{2\gamma^{-\frac{3}{2}}}{3} = \frac{2}{3\sqrt{3\frac{\gamma^3}{3}}} = \frac{2}{3\sqrt{3\alpha}}$$

that corresponds to the previous expression.

**Lemma 6.** *For any real value of $\rho$, any positive value of $\delta > 0$ and for every real value of $x_1$ and $x_2$ and any nonegative value of $a > 0$ the inequality*

$$\rho x_1^{\frac{1}{3}} x_2 - \delta^2 \frac{|x_2|^2}{|x_1|^{\frac{1}{3}}} \leq \frac{a}{\delta^2} |x_1| - \delta \nu |x_2|^{\frac{3}{2}} \tag{28}$$

*is satisfied if and only if*

$$\nu \leq b(a; \rho) \ ,$$

*where the function $b(a; \rho)$ is defined in* (4).

## *B. Negative definiteness of $\dot{V}$*

Using inequalities (28), (25) and (27) it follows

$$\begin{aligned}
\dot{V} \leq & -q_{11} |x_1| + \frac{3}{2} \frac{a}{\gamma_{12}} |x_1| - \sqrt{\frac{2}{3}} \gamma_{12} b(a; \rho) |x_2|^{\frac{3}{2}} + \\
& + \gamma_{23} k_2 \alpha |x_1| + \beta(\alpha; \lambda, \mu(t)) \gamma_{23} k_2 |x_3|^3 - \gamma_{23} |x_3|^3 \\
& + 2(|\gamma_2| + |\gamma_{23}(k_3 + \Delta)|) \left( \frac{2\beta_{23}^{\frac{3}{2}}}{3} |x_2|^{\frac{3}{2}} + \frac{\beta_{23}^{-3}}{3} |x_3|^3 \right) ,
\end{aligned}$$

where $b(a; \rho)$ is given by (4), $\rho = q_{12}$, $\beta(\alpha; \lambda, \mu(t))$ is given by (6), $\lambda$ and $\mu$ are defined in (16) and (17), respectively. We have assumed that inequalities (10), (11) and (12) are satisfied.

Therefore

$$\begin{aligned}
\dot{V} \leq & -\left( \left( \tfrac{4}{3}\gamma_1 k_1 - \gamma_{12} k_2 \right) - \tfrac{3}{2} \tfrac{a}{\gamma_{12}} - \gamma_{23} k_2 \alpha \right) |x_1| + \\
& - \left( \sqrt{\tfrac{2}{3}} \gamma_{12} b(a) - 2(|\gamma_2| + |\gamma_{23}(k_3 + \Delta)|) \tfrac{2\beta_{23}^{\frac{3}{2}}}{3} \right) |x_2|^{\frac{3}{2}} \\
& - \left( \gamma_{23} - \gamma_{23} k_2 \beta(\alpha) - 2(|\gamma_2| + |\gamma_{23}(k_3 + \Delta)|) \tfrac{\beta_{23}^{-3}}{3} \right) |x_3|^3 \ ,
\end{aligned}$$

and $\dot{V}$ is negative definite if, for some positive values of the gains and some values of $\alpha > 0$, $\beta_{23} > 0$, $a > 0$ the following inequalities

$$\left(\frac{4}{3}\gamma_1 k_1 - \gamma_{12} k_2\right) - \frac{3}{2}\frac{a}{\gamma_{12}} - \gamma_{23} k_2 \alpha > 0 \tag{29a}$$

$$\sqrt{\frac{2}{3}\gamma_{12}} b\left(a\right) - 2\left(\left|\gamma_2\right| + \left|\gamma_{23}\left(k_3 + \Delta\right)\right|\right) \frac{2\beta_{23}^{\frac{3}{2}}}{3} > 0 \tag{29b}$$

$$\gamma_{23} - \gamma_{23} k_2 \beta\left(\alpha\right) - 2\left(\left|\gamma_2\right| + \left|\gamma_{23}\left(k_3 + \Delta\right)\right|\right) \frac{\beta_{23}^{-3}}{3} > 0 \tag{29c}$$

are fulfilled. From (29b) it follows that $b\left(a;\rho\right) > 0$ has to be satisfied, what we will assume henceforth.

We eliminate from (29) the dummy variables $a$ and $\beta_{23}$.

**Elimination of $\beta_{23}$**

Conditions (29b), (29c) can be writen as

$$\frac{3}{2^2}\sqrt{\frac{2}{3}\gamma_{12}} \frac{b\left(a;\rho\right)}{\left(\left|\gamma_2\right| + \left|\gamma_{23}\left(k_3 + \Delta\right)\right|\right)} > \beta_{23}^{\frac{3}{2}}$$

$$\beta_{23}^{\frac{3}{2}} > \sqrt{\frac{2}{3}\frac{\left(\left|\gamma_2\right| + \left|\gamma_{23}\left(k_3 + \Delta\right)\right|\right)}{\gamma_{23} - \gamma_{23} k_2 \beta\left(\alpha\right)}}$$

where it is required that $\gamma_{23} - \gamma_{23} k_2 \beta\left(\alpha\right) > 0$.

There exists such a $\beta_{23}$ if

$$b\left(a;\rho\right) > \frac{2^2}{3}\sqrt{\frac{\left(\left|\gamma_2\right| + \left|\gamma_{23}\left(k_3 + \Delta\right)\right|\right)^3}{\gamma_{12}\left(\gamma_{23} - \gamma_{23} k_2 \beta\left(\alpha\right)\right)}} > 0 \tag{30}$$

**Elimination of $a$**

Using (29a) one obtains

$$\frac{2}{3}\left(\frac{4}{3}\gamma_1 k_1 - \gamma_{12} k_2 - \gamma_{23} k_2 \alpha\right)\gamma_{12} > a\,.$$

Due to the monotonicity of the function $b$ with respect to $a$, and in view of (30), such an $a$ exists if (13) is satisfied.

### *C. Differential inequality for $V$*

We want to show that, given that $V$ is p.d. and $\dot{V}$ is n.d., the differential inequality (18) is satisfied. Due to the homogeneity of $V$ and $\dot{V}$, with degrees $\delta_V = 4$ and $\delta_{\dot{V}} = 3$, respectively, it follows that the function

$$W\left(x\right) = \frac{-\dot{V}\left(x\right)}{V^{\frac{3}{4}}\left(x\right)}$$

is homogeneous of degree $\rho_W = 0$, and therefore, since $W\left(x\right) = W\left(k^3 x_1,\, k^2 x_2,\, k x_3\right)$ for every $k > 0$, all values of the function are taken on the unity homogeneous ball, i.e. $B_h = \left\{x \in \mathbb{R}^3 \mid \left|x_1\right|^{\frac{2}{3}} + \left|x_2\right| + \left|x_3\right|^2 = 1\right\}$. On $B_h$ the function $V$ is continuous and is different from zero. On $B_h$ the function $-\dot{V}$ is different from zero, and it is continuous almost everywhere, except on the points $\left\{x_1 = 0\right\}$. However, on these discontinuity points it becomes $+\infty$. Therefore, $W\left(x\right)$ has a (positive) minimum, that can be calculated by

$$\kappa = \min_{x \in B_h} W\left(x\right)\,.$$

This implies that

$$\frac{-\dot{V}\left(x\right)}{V^{\frac{3}{4}}\left(x\right)} \geq \kappa \Longrightarrow \dot{V}\left(x\right) \leq -\kappa V^{\frac{3}{4}}\left(x\right)\,.$$

From this differential inequality it follows immediately that the trajectories converge in finite time to zero. The solution to the differential equation$\dot{v}\left(t\right) = -\kappa v\left(t\right)$ is given by $v^{\frac{1}{4}}\left(t\right) = v^{\frac{1}{4}}\left(0\right) - \frac{1}{4}\kappa t$. From the comparison lemma it easily follows that the convergence time can be estimated as in (19).

To prove the Corollary one has just to apply the transformations $\left(k_1,\, k_2,\, k_3\right) \rightarrow \left(L k_1,\, L^2 k_2,\, L^3 k_3\right)$ and $\left(\gamma_1,\, \gamma_{12},\, \gamma_2,\, \gamma_{23},\, \gamma_3\right) \rightarrow \left(L^{-4}\gamma_1,\, L^{-5}\gamma_{12},\, L^{-6}\gamma_2,\, L^{-9}\gamma_{23},\, L^{-12}\gamma_3\right)$ in all the inequalities. The result for $\Delta = 0$ is immediate, since the inequalities remain invariant under this transformation.

## V. Conclusions

In this paper we provide, for the first time in the literature, a Lyapunov function for Levant's exact and robust second order differentiator. Although the Lyapunov function presented here is quadratic its derivative is not, and it is also a discontinuous function. Some new inequalities were introduced in the paper in order to show the negative definiteness of the derivative of this Lyapunov function. Technically, this is the main contribution of the paper. This technique can be also used for non quadratic Lyapunov functions for the same differentiator, and it seems to be possible to generalize the results for higher order differentiators, what is the topic of future research.

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